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Vibration Analysis of a Two-Stage Spur-Geared Mechanical System

By Prof. Shuting Li

In order to be able to accurately analyze the vibration of geared mechanical systems, the studies on constructing a vibration analysis method and developing software were begun very early by combining the theoretical modal method and the Duhamel integral method. But these studies could not be completed for a long time because stable results could not be obtained in some cases. Through many years of effort, the cause of the unstable results was found finally, and it succeeded in developing software that can obtain stable results in all cases. This paper explains the technical difficulties that must be overcome when analyzing the vibration of geared mechanical systems at the first. Then the flowchart used for numerical analysis of gear vibration and software development is introduced. In order to verify the validity of the developed software, a vibration analysis on a two-stage spur gear system is performed. Since the calculated results are almost consistent with the experimental results, it can be said that a reliable way is found to analyze geared mechanical systems. The results analyzed by the newly developed software provide some new insights into the phenomenon of gear vibration.

Key Words : Gear, Spur Gear, Geared Machines, Modeling, Vibration Analysis, Acceleration

1. Introduction

Gears are used to reduce or increase the speed of various rotating machines and are a crucial component of such machinery. Since gear vibration and noise directly affect the vibration and noise of rotating machines, reducing gear vibration and noise directly contributes to reducing the vibration and noise of rotating machines. To reduce gear vibration and noise, it is necessary to accurately analyze gear vibration theoretically; however, due to the numerous factors affecting gear vibration, the problem of gear vibration analysis has not yet been fully resolved.

Many researchers have undertaken studies to enable the precise analysis of vibrations in mechanical systems that use gears. The author also began developing a gear vibration analysis method and dedicated software in 1987 by combining the theoretical mode method with the Duhamel integration method; however, since stable results could not always be obtained depending on the subject of analysis, the author struggled with this research for a long time. After many years of effort, the author finally identified the factors causing unstable results and succeeded in developing software capable of producing stable results for all cases.

In this study, the author explains the technical challenges that must be overcome when analyzing the vibration of mechanical systems using gears, while introducing the process of numerical analysis of gear vibration and software development through the combined use of the theoretical mode method and the Duhamel integration method. To verify the validity of the developed software, we performed a vibration analysis on a two-stage spur gear reduction mechanism and compared the analytical results with experimental data. Since the analytical results were in close agreement with the experimental results, we also performed vibration analyses on mechanical systems such as power-absorbing gear test machines and power-circulating gear test machines, and in all cases, valid results were obtained. Furthermore, upon analyzing these results, we discovered new insights regarding gear vibration.

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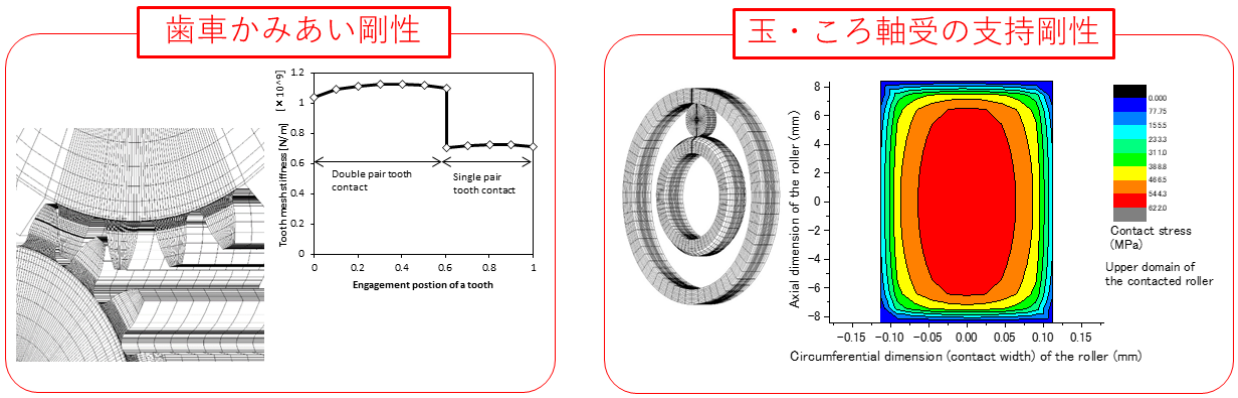
2. Technical Challenges in Gear Vibration Analysis

In order to analyze gear vibrations, the following technical challenges must first be overcome: (1) Theoretical analysis of tooth meshing stiffness and bearing support stiffness; (2) Handling tooth separation phenomena in vibration analysis; (3) The effects of tooth machining errors, assembly errors, and tooth profile modification on vibrations; (4) Methods for solving nonlinear equations of motion; (5) Determining structural vibration responses when analyzing the structural vibration of thin-walled gears (analysis techniques for large-scale, multi-degree-of-freedom structural vibration); (6) Determining vibration responses when different vibration modes overlap or have closely spaced frequencies. Overcoming these challenges requires many years of research and continuous effort, which demonstrates that gear vibration cannot be easily analyzed.

To analyze gear vibration, it is first necessary to determine the meshing stiffness of a pair of gears. Since meshing stiffness varies depending on the meshing position (rotational angle) and is also influenced by tooth geometry (involute curve, shifting coefficient, helical angle, etc.), it is well known that meshing stiffness generally cannot be easily determined. In this study, the meshing stiffness of a pair of spur gears was treated as a contact problem between a pair of elastic bodies and was determined by developed 3D, FEM software, as shown in Figure 1(a)[(1)]. The results are also shown in Figure 1(a).

To investigate the effect of bearing support stiffness on gear vibration, it is necessary to theoretically determine the bearing's support stiffness; however, since there is no simple method for calculating this stiffness, in this study, we developed specialized 3D, FEM software through many years of research and effort, enabling the calculation of support stiffness for deep-groove ball bearings, angular contact ball bearings, cylindrical roller bearings, and tapered roller bearings[(2)(3)(8)]. Figure 1(b) shows the FEM model for contact analysis of a cylindrical roller bearing and the contact pressure distribution on the roller surface. The support stiffness values are presented in References [(2)(3)(8)].



(a) FEM model and Tooth mesh stiffness　　(b) FEM model and pressure of rolling bearings

Fig.1 Contact stiffness analyses of gears and bearings using 3D, FEM.

Since numerous studies have been conducted on the effects of gear manufacturing and assembly errors, as well as tooth profile modification, on tooth separation and vibration, we will omit a detailed explanation of this topic here; instead, we will briefly introduce the new findings from this study. In conventional approaches, gear meshing stiffness is represented by a single spring, as shown in Figure 2(a). The effects of manufacturing errors and tooth profile modifications on tooth separation and vibration were analyzed by attaching a clearance element (see Figure 2(a)) to the tip of this spring. However, since I believed that a single spring could not adequately represent the complex tooth-tooth contact conditions, I concluded

that these conditions should be modeled using multiple springs, as shown in Figure 2(b). Consequently, I proposed a surface contact model featuring multiple springs, as illustrated in Figure 2(b)[4]. Using this surface contact model, I analyzed the static contact problem of a pair of spur gears[1] and also developed vibration analysis software[4]. However, since the validity of the results obtained using this surface contact model has not yet been verified, this paper first presents the results of the analysis using a single spring. I plan to publish the results of the analysis using multiple springs in the near future.

As shown in Figure 2(a), when meshing stiffness is represented by a single spring, the meshing stiffness varies nonlinearly depending on the meshing position; consequently, the equations of motion derived for gear vibration analysis also become nonlinear. To enable the solution of these nonlinear equations of motion, we linearized them[5].

In the development of aerospace machinery, the thickness of gear rims and webs must be reduced to achieve weight reduction. This thinning causes structural vibrations in the rims and webs; therefore, 3D-FEM is required to analyze these structural vibrations. In other words, it is necessary to establish a method for calculating large-scale (many degrees of freedom) structural vibration responses. Furthermore, when different vibration modes overlap or have closely spaced frequencies[6], conventional methods can no longer be used to determine the vibration response, making it necessary to develop new methods. Although research on vibration modes and frequency analysis[6] has been conducted regarding the vibration of thin-walled gears, and experimental studies on vibration phenomena have also been performed[6], there has been very little research on vibration response analysis[7], and the validity of the analytical results has not been thoroughly examined; therefore, this issue remains unresolved.

(a) Conventional one-spring model　　(b) Newly proposed multiple-springs model

Fig.2 Contact models of contact teeth used for vibration analysis of gears

3. Theory of Gear Vibration Analysis and Development of Dedicated Software

To determine the vibration response of mechanical systems using gears, we developed specialized software based on the flowchart shown in Figure 3. However, when analyzing vibration responses with this software, we encountered a problem where the results became unstable depending on the subject of analysis. Therefore, we worked to resolve this issue through many years of theoretical and experimental research. As a result, we gained a wealth of expertise, identified the causes of the instability, and discovered methods to stabilize the analysis results. Currently, we are performing vibration analyses on mechanical systems such as power-absorbing gear test machines, power-circulating gear test machines, and two-stage gear reducers. Since stable results have been obtained in all analyses, we present below the vibration analysis results for a two-stage spur gear reducer as an example. Due to space limitations, the vibration analysis results for other mechanical systems are omitted here.

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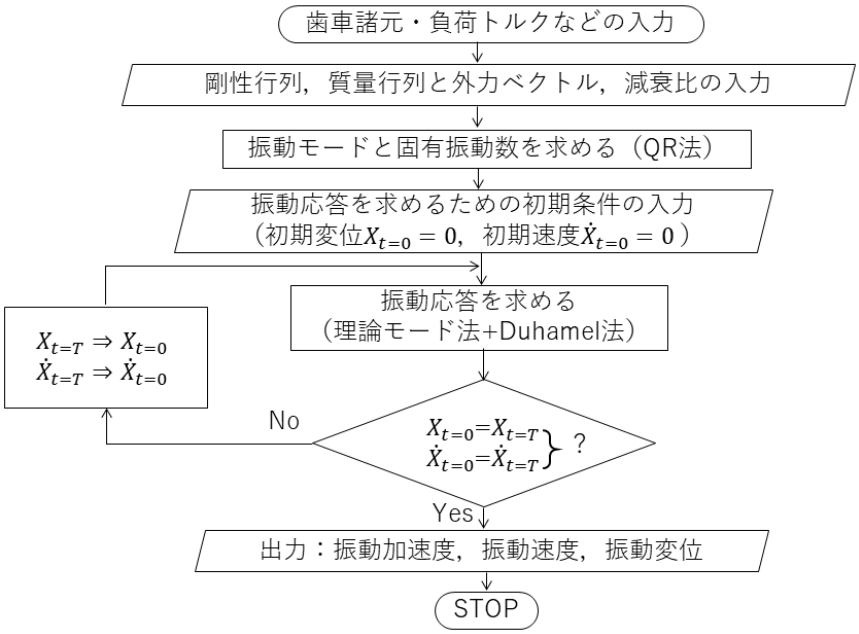

Fig.3 Flow-chart of software development used for gear vibration analysis

4. Examples of Applications for the Vibration Analysis Software

To verify the validity of the developed software, we first performed a vibration analysis modeling of the two-stage spur gear reduction mechanism shown in Figure 4. The results are shown in Figure 5. We then used the model shown in Figure 5 to derive the equations of motion using the *Lagrangian* method. Finally, by solving the derived equations of motion using the workflow shown in Figure 3 and the developed software, we obtained the vibrational displacement, vibrational velocity, and vibrational acceleration of each mass point.

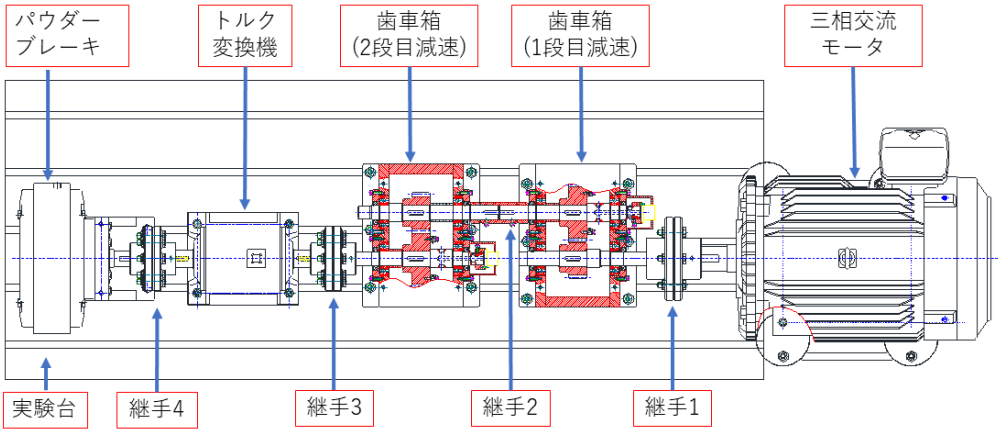

Fig.4 A two-stage geared mechanical system

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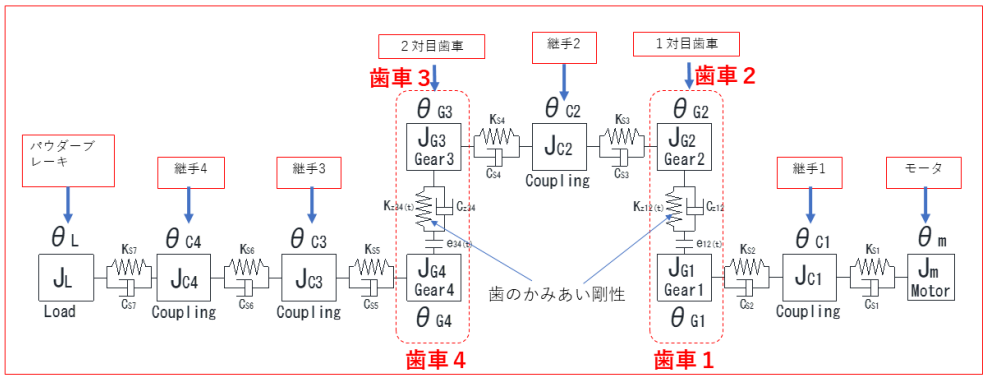
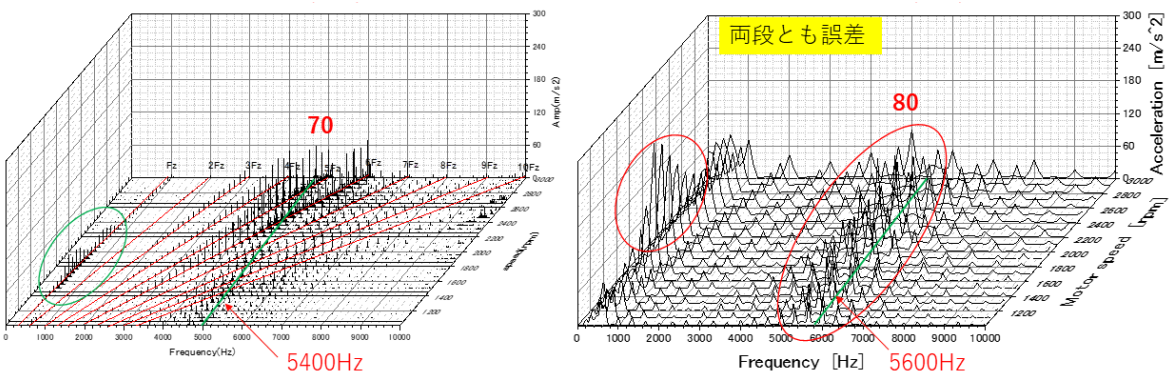


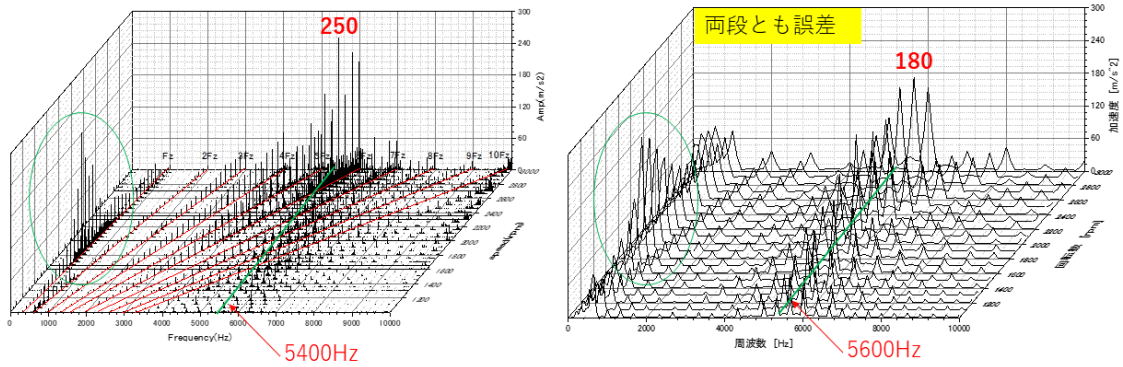
Fig.5 Modeling for vibration analysis of the geared mechanical system

Figures 6 and 7 show Campbell diagram created using the rotational vibration acceleration responses of Gear 4. Figures 6(a) and 7(a) are Campbell diagram of the vibration acceleration signals measured experimentally, while Figures 6(b) and 7(b) are Campbell diagram of the vibration acceleration signals analyzed using software. The analysis accounted for tooth profile errors and pitch errors in the two pairs of gears. Figure 6 shows a comparison at a load torque of 110 Nm, while Figure 7 shows a comparison at a load torque of 306 Nm. A comparison of Figure 6(a) and Figure 6(b), as well as Figure 7(a) and Figure 7(b), reveals that the results analyzed by the software are in close agreement with the experimentally measured results.



(a) Measured accelerations　　(b) Calculated accelerations

Fig.6 Campbell diagram comparison of gear accelerations (Torque=110Nm)



(a) Measured accelerations　　(b) Calculated accelerations

Fig.7 Campbell diagram comparison of gear vibration accelerations (Torque=306Nm)

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7. Conclusion

① Based on the theoretical mode method and the Duhamel integration method, we developed specialized vibration analysis software while establishing a vibration analysis method for mechanical systems using gears.

② Using the software we developed, we analyzed the vibrations of a two-stage gear reduction mechanism and compared the analysis results with experimental data; the analysis results were found to be valid. We also obtained valid results for the vibrations of other mechanical systems that use gears.

③ Through many years of theoretical and experimental research, we have acquired a wealth of expertise regarding gear vibrations and have determined that the vibrations of mechanical systems using gears can be analyzed by combining the theoretical mode method with the Duhamel integration method.

④ An analysis of the gear vibration results obtained using the software we developed revealed that the fundamental component f_z of the tooth meshing frequency was caused by tooth machining errors. We were also able to successfully analyze the tooth resonance frequency.

⑤ Since we have developed a new method for gear vibration analysis, this method is expected to be applied in the future to the vibration analysis of complex mechanical systems such as helicopters and wind turbines.

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